

Currents carried by the subgradient graphs of semi-convex functions and applications to Hessian measure

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Abstract: In this paper we study integer multiplicity rectifiable currents carried by the subgradient (subdifferential) graphs of semi-convex functions on a n -dimensional convex domain, and show a weak continuity theorem with respect to pointwise convergence for such currents. As an application, the k -Hessian measures are calculated by a different method in terms of currents.

Key words: Semi-convex function; subgradient; Cartesian current; Hessian measure.

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1 Introduction and main results

This paper is devoted to the study of some properties and applications of the subgradient (or subdifferential) graphs of semi-convex functions defined on a convex domain $\Omega \subset \mathbb{R}^n$. Let \mathcal{L}^n and \mathcal{H}^n be the n -dimensional Lebesgue and Hausdorff measure as usual.

Following the Cartesian current theory, Giaquinta-Modica-Souček [10] introduced a class of functions $u \in L^1(\Omega, \mathbb{R}^n)$, named $\mathcal{A}^1(\Omega, \mathbb{R}^n)$, such that u is approximately differentiable a.e. and all minors of the Jacobian matrix Du are summable in Ω . For $u \in \mathcal{A}^1(\Omega, \mathbb{R}^n)$, it is well defined an integer multiplicity (i.m.) rectifiable current G_u carried by the rectifiable graph of u . More precisely,

$$G_u = \tau(\mathcal{G}_{u,\Omega}, 1, \xi_u).$$

The unit n -vector $\xi_u(x, u(x)) = \frac{M(Du(x))}{|M(Du(x))|}$ given at each point $(x, u(x)) \in \mathcal{G}_{u,\Omega}$ provides an orientation to the approximate tangent space $Tan^n(\mathcal{G}_{u,\Omega}, x)$. And the rectifiable graph of u is given by

$$\mathcal{G}_{u,\Omega} = \{(x, u(x)) \mid x \in \mathcal{L}_u \cap A_D(u) \cap \Omega\},$$

where \mathcal{L}_u is the set of Lebesgue points and $A_D(u)$ is the set of approximate differentiability points of u , for more details see [10, Vol. I, Sect. 3.2.1]. Moreover the area of $\mathcal{G}_{u,\Omega}$ is equal to the mass of G_u , i.e.,

$$\mathcal{H}^n(\Gamma_{u,\Omega}) = \int_{\Omega} |M(Du(x))| dx = \mathbf{M}(G_u).$$

In the sequel we study the properties of subgradient ∂w of a semi-convex function w in terms of Cartesian currents. The initial motivation of our work is the following: Alberti-Ambrosio [1] studied some analytical properties of monotone set-valued maps defined on \mathbb{R}^n , defined n -currents on $\mathbb{R}^n \times \mathbb{R}^n$ for maximal monotone maps on \mathbb{R}^n and gave some continuity and approximation results

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for such currents. As we know, an important class of maximal monotone maps is represented by the subgradients of convex functions. A natural problem is raised whether we can extend the definitions and results of current for $\mathcal{A}^1(\Omega, \mathbb{R}^n)$ and maximal monotone maps to the subgradients of semi-convex functions (denoted by $W(\Omega)$). Here we try to discuss this problem. More precisely, we define an i.m. rectifiable current $G_{\partial w}$ carried by the subgradient graph (denoted by $\Gamma_{\partial w, \Omega}$) of w such that the current has zero boundary and the orientation in “nonvertical parts” is consistent with the one given in the class $\mathcal{A}^1(\Omega, \mathbb{R}^n)$ a.e. The following is our first main result.

Theorem 1.1. *If $w \in W(\Omega)$ and a single-valued map $f : \Omega \rightarrow \mathbb{R}^n$ such that $f(x) \subset \partial w(x)$ for any $x \in \Omega$. Then there exists an i.m. rectifiable current $G_{\partial w}$ such that:*

- (i) $G_{\partial w} = \tau(\Gamma_{\partial w, \Omega}, 1, \xi) \in \mathcal{R}_{n, \text{loc}}(\Omega \times \mathbb{R}^n)$.
- (ii) $\xi(x, f(x)) = \frac{M(Df(x))}{|M(Df(x))|}$ for \mathcal{L}^n a.e. $x \in \Omega$, and $G_{Dw} = G_{\partial w}$ when $w \in C^2$.
- (iii) $\partial G_{\partial w} \llcorner \Omega \times \mathbb{R}^n = 0$ and $\mathcal{H}^n(\Gamma_{\partial w, B}) = \mathbf{M}_{B \times \mathbb{R}^n}(G_{\partial w})$ for any Borel set $B \subset \subset \Omega$.

It turns out that $G_{\partial w}$ is the push-forward of an i.m. rectifiable current under a rotation transformation. However this quantity is well-defined by the fact that the current is independent of the choice of rotation transformations. For more details, see Theorem 3.4.

An important problem is to characterize the Cartesian currents $T \in D_n(\Omega \times \mathbb{R}^n)$ for which there is a sequence of smooth maps $u_k : \Omega \rightarrow \mathbb{R}^n$ such that

$$G_{u_k} \rightharpoonup T, \quad \mathbf{M}(G_{u_k}) \rightarrow \mathbf{M}(T).$$

Such a question is connected with the problem of relaxation of the area integral for nonparametric graphs which is discussed by Giaquinta-Modica-Souček in [9], [10, Vol. II, Ch. 6]. Then Mucci made efforts to investigate the problem and showed that functions satisfying some certain conditions can be approximated weakly in the sense of currents and in area by graphs of smooth map (see [15, 16, 17, 18]). It motivates us to focus on the question whether $G_{\partial w}$ can be approximated by smooth currents. In view of this, we show the following weak continuity theorem in current sense for semi-convex functions.

Theorem 1.2 (Weak continuity theorem). *If $w, w_k \in W(\Omega, c)$ such that $w_k \rightarrow w$ pointwise in Ω as $k \rightarrow \infty$, then $G_{\partial w_k} \rightharpoonup G_{\partial w}$ in $\mathcal{D}^n(\Omega \times \mathbb{R}^n)$.*

According to Theorem 1.2, the current $G_{\partial w}$ carried by the graph of subgradient of a semi-convex function w is Lagrangian, which one can think of as meaning “weakly curl-free” (see [12] and below for more details). Moreover, the current can be weakly approximated by smooth currents if w is also Lipschitz.

As an application of the current defined as above, a method is proposed to calculate the k -Hessian measures for semi-convex functions. Trudinger-Wang [20, 21] introduced the notion of k -Hessian measures as Borel measures associated to k -convex functions to study weak solutions of some elliptic partial differential equations. Then Colesanti-Hug [6] proved that the k -Hessian measures of semi-convex functions can be defined as coefficients of a local Steiner type formula, and pointed out the equivalence of the two definitions in the class of semi-convex functions. Now, a formula for the k -Hessian measures in terms of currents for semi-convex functions is as follow.

Theorem 1.3. *Let $w \in W(\Omega)$ and $G_{\partial w} := \tau(\Gamma_{\partial w, \Omega}, 1, \xi)$, then for every Borel subset $B \subset \subset \Omega$, the k -Hessian measure of w can be written as*

$$F_k(w, B) = \sum_{|\alpha|=k} \sigma(\alpha, \bar{\alpha}) \int_{\Gamma(\partial w, B)} \xi^{\alpha \bar{\alpha}}(x, y) d\mathcal{H}^n(x, y) \quad k = 0, 1, \dots, n.$$

In particular, $F_0(w, B) = \mathcal{L}^n(B)$; $F_n(w, B) = \mathcal{L}^n(\partial w(B))$ if w is convex.

This paper is organized as follows. Some facts and notion about semi-convex functions, set-valued maps and Cartesian currents are given in Section 2. Then we prove Theorem 1.1 in Section 3. In Section 4 we show the weak continuity theorems for semi-convex functions. Finally in Section 5 we give a different formula for k -Hessian measures of semi-convex functions.

2 Preliminaries

This section reviews some notion and basic facts about semi-convex functions, set-valued maps and Cartesian currents. For more details, see [11, 8, 3, 12, 10, 19].

Definition 2.1. A real-valued function $w : \Omega \rightarrow \mathbb{R}$ is called semi-convex if there exists $c \geq 0$ such that the function $w(x) + \frac{c}{2}|x|^2$ is convex in Ω .

For a semi-convex function w in Ω , the semiconvexity modulus of w is defined by

$$sc(w, \Omega) := \inf\{c \mid w + \frac{c}{2}|x|^2 \text{ is convex in } \Omega\}.$$

We set the class of semi-convex functions by $W(\Omega)$ and $W(\Omega, c) := \{w \in W(\Omega) \mid sc(w, \Omega) \leq c\}$.

Examples 2.2. (i) The simplest examples of semi-convex functions are convex functions.

(ii) The viscosity solutions of some Hamiltonian-Jacobian equations are semi-concave, the class of functions u such that $-u \in W(\Omega)$, see Lions [13].

(iii) The distance function from a closed subset K of a n -dimensional Riemannian manifold (M, g) are locally semi-concave in $M \setminus K$, see Mantegazza-Mennucci[14].

We denote by $P(\mathbb{R}^n)$ the collection of subsets of \mathbb{R}^n , $P_0(\mathbb{R}^n) := P(\mathbb{R}^n) \setminus \{\emptyset\}$ and I both the identity map on \mathbb{R}^n and the $n \times n$ identity matrix. Given a set-valued map $F : \Omega \rightarrow P(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$, we set

$$\begin{aligned} \text{graph of } F, \Gamma_{F,A} &:= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid y \in F(x), x \in \Omega\}, \\ \text{image of } F, F(A) &:= \{y \in \mathbb{R}^n \mid y \in F(x), x \in A\}. \end{aligned}$$

Definition 2.3. A set-valued map $F : \Omega \rightarrow P(\mathbb{R}^n)$ is monotone in Ω if its graph is monotone, i.e.,

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$$

for all $(x_i, y_i) \in \Gamma_{F, \Omega}$ with $i = 1, 2$. A monotone map F is maximal in Ω if there is no other monotone set-valued map in Ω whose graph strictly contains the graph of F .

Definition 2.4. Let $w : \Omega \rightarrow \mathbb{R}$, the subgradient (subdifferential) $\partial w(x)$ of w at x is defined by

$$\partial w(x) = \{p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{w(y) - w(x) - \langle p, y - x \rangle}{|y - x|} \geq 0\}.$$

We also recall some elementary properties of subgradient of semi-convex functions.

Proposition 2.5. Let $w \in W(\Omega, c)$.

- (i) w is locally Lipschitz in Ω and the image $\partial w(A)$ is bounded for any bounded subset $A \subset \subset \Omega$.
- (ii) $\partial w(x)$ is a nonempty, closed and convex set for any $x \in \Omega$. Moreover,

$$y \in \partial w(x) \Leftrightarrow w(z) \geq w(x) + (z - x) \cdot y - \frac{c}{2}|z - x|^2 \text{ for all } z \in \Omega.$$

- (iii) w has a second derivative for \mathcal{L}^n a.e. on Ω . Moreover, ∂w is a maximal semi-monotone map in Ω , i.e., $\partial w + cI$ is maximal monotone map in Ω .
- (iv) $(sI + \partial w)(\Omega) = \mathbb{R}^n$ if $\Omega = \mathbb{R}^n$, where $s > c$.

Proof. If we take $v(x) = w(x) + \frac{c}{2}|x|^2$, then clearly v is convex. Now (i), (ii), (iii) and (iv) are immediately followed from [7, Proposition 2.1.5, 2.4.1], [2, Proposition 2.1], [1, Theorem 3.2] and [6, Theorem 3.5.8], respectively. \square

In order to obtain the main results, we introduce the definition of geometrically derivatives for set-valued maps from the choice of tangent cones to the graphs (see [3]).

Definition 2.6. Let a set-valued map $F : \Omega \rightarrow P(\mathbb{R}^n)$. The contingent derivative $DF(x, y)$ of F at $(x, y) \in \Gamma_{F, \Omega}$ is the set-valued map from \mathbb{R}^n to \mathbb{R}^n defined by

$$\Gamma_{DF(x, y), \mathbb{R}^n} := \left\{ (p, q) \mid \liminf_{h \rightarrow 0^+} \frac{d((x + hp, y + hq), \Gamma(F, \Omega))}{h} = 0 \right\}.$$

It is very convenient to have the following characterization of contingent derivatives in terms of sequences: $q \in DF(x, y)(p)$ if and only if there exist $h_m \rightarrow 0^+$, $p_m \rightarrow p$ and $q_m \rightarrow q$ as $m \rightarrow \infty$ such that $y + h_m q_m \in F(x + h_m p_m)$.

Proposition 2.7. If $F := f$ is a single-valued map and differentiable at x , then $Df(x, f(x))(p) = Df(x)p$ for any $p \in \mathbb{R}^n$.

Proof. The proof can be seen in [3, Proposition 5.1.2]. \square

Next we recall some notation and facts about currents and Geometric Measure Theory.

For integers $n, N \geq 2$, we shall use the standard notation for ordered multi-indices

$$I(k, n) := \{\alpha = (\alpha_1, \dots, \alpha_k) \mid \alpha_i \text{ integers}, 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}.$$

Set $I(0, n) = \{0\}$ and $|\alpha| = k$ if $\alpha \in I(k, n)$. If $\alpha \in I(k, n)$, $k = 0, 1, \dots, n$, $\bar{\alpha}$ is the element in $I(n - k, n)$ which complements α in $\{1, 2, \dots, n\}$ in the natural increasing order. So $\bar{0} = \{1, 2, \dots, n\}$. For $i \in \alpha$, $\alpha - i$ means the multi-index of length $k - 1$ obtained by removing i from α .

Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be $n \times n$ matrixes. Given two ordered multi-indices with $\beta \in I(k, n), \alpha \in I(n - k, n)$, then A_{α}^{β} denotes the $k \times k$ -submatrix of A with rows $(\beta_1, \dots, \beta_k)$ and columns $(\alpha_1, \dots, \alpha_k)$. Its determinant will be denoted by

$$M_{\alpha}^{\beta}(A) := \det A_{\alpha}^{\beta}.$$

We shall set

$$M_{\alpha\beta}(A, B) := \det(C), \quad \text{where } c_{ij} = \begin{cases} a_{ij}, & i \in \alpha, \\ b_{ij}, & i \in \beta. \end{cases}$$

The adjoint of A_{α}^{β} is defined by the formula

$$(\text{adj } A_{\alpha}^{\beta})_j^i := \sigma(i, \beta - i) \sigma(j, \alpha - j) \det A_{\alpha - j}^{\beta - i} \quad i \in \beta, j \in \alpha,$$

where $\sigma(\cdot, \cdot)$ is the sign of the permutation with reorders. So Laplace formulas can be written as

$$M_{\alpha}^{\beta}(A) = \sum_{j \in \alpha} a_{ij} (\text{adj } A_{\alpha}^{\beta})_j^i.$$

Let U be a open set in \mathbb{R}^n , we denote by $\mathcal{D}^k(U)$ the spaces of compactly supported k -form in U . The dual space to $\mathcal{D}^k(U)$ is the class of k -currents $\mathcal{D}_k(U)$. For any open set $V \subset\subset U$ the mass of a current $T \in \mathcal{D}_k(U)$ in V is

$$\mathbf{M}_V(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U), \text{ spt } \omega \subset V, \|\omega\| \leq 1\},$$

and $\mathbf{M}(T) := \mathbf{M}_U(T)$ denote the mass of T .

A current $T = \tau(\mathcal{M}, \theta, \xi) \in \mathcal{D}_k(U)$, is called an integer multiplicity rectifiable k -current (briefly i.m. rectifiable current) if it can be expressed

$$T(\omega) = \int_{\mathcal{M}} \langle \omega(x), \xi(x) \rangle \theta(x) d\mathcal{H}^k(x), \quad \omega \in \mathcal{D}^k(U),$$

where \mathcal{M} is an \mathcal{H}^k -measurable countably k -rectifiable subset of U , θ is a locally \mathcal{H}^k -measurable positive integer-valued function, and $\xi : \mathcal{M} \rightarrow \wedge_k(\mathbb{R}^n)$ is a \mathcal{H}^k -measurable function such that for \mathcal{H}^n -a.e. point $x \in \mathcal{M}$, $\xi(x)$ provides an orientation to the approximate tangent spaces $Tan^k(\mathcal{M}, x)$. θ is called the multiplicity and ξ is called the orientation for T . The i.m. rectifiable k -currents in $\mathcal{D}_k(U)$ is denote by $\mathcal{R}_k(U)$ if T has finite mass, and $T \in \mathcal{R}_{k,loc}(U)$ if T has local finite mass.

Let $T = \tau(\mathcal{M}, \theta, \xi) \in \mathcal{R}_k(U)$, and $f : U \rightarrow V \subset \mathbb{R}^n$ be a Lipschitz map such that $f|_{\text{spt } T}$ is proper. Then the push-forward of T under f turns out to be an i.m rectifiable k -current which can be explicitly written as (see [8, 10])

$$\begin{aligned} f_{\#}T(\omega) &= \int_{\mathcal{M}} \langle \omega(f(x)), (\wedge_k d^{\mathcal{M}} f) \xi(x) \rangle \theta(x) d\mathcal{H}^k(x) \\ &= \int_{f(\mathcal{M})} \langle \omega(y), \sum_{x \in f^{-1}(y) \cap \mathcal{M}_+} \theta(x) \frac{(\wedge_k d^{\mathcal{M}} f) \xi(x)}{|(\wedge_k d^{\mathcal{M}} f) \xi(x)|} \rangle d\mathcal{H}^k(y), \end{aligned} \tag{2.1}$$

where

$$\mathcal{M}_+ = \{x \in \mathcal{M} \mid J_f^{\mathcal{M}}(x) = |(\wedge_k d^{\mathcal{M}} f) \xi(x)| > 0\}.$$

$T = \tau(\mathcal{M}, \theta, \xi) \in \mathcal{R}_{n,loc}(\Omega \times \mathbb{R}^n)$ ($n \geq 2$) is called Lagrangian if for \mathcal{H}^n a.e. $(x, y) \in \mathcal{M}$, the approximate tangent space $Tan^n(\mathcal{M}, (x, y))$ satisfies

$$\langle \phi, \tau_1 \wedge \tau_2 \rangle = 0 \text{ for any two vectors } \tau_1, \tau_2 \text{ tangent to } Tan^k(\mathcal{M}, (x, y))$$

where $\phi := \sum_{i=1}^{\infty} dx^i \wedge dy^i$. One can check that T is Lagrangian if and only if

$$T(\phi \wedge \eta) = 0 \text{ for any } \eta \in \mathcal{D}_{n-2}(\Omega \times \mathbb{R}^n).$$

3 The proof of Theorem 1.1

For $u \in \mathcal{A}^1(\Omega, \mathbb{R}^N)$, the current $G_u = \tau(\mathcal{G}_{u,\Omega}, 1, \xi_u) \in \mathcal{R}_n(\Omega \times \mathbb{R}^N)$ is defined for $\omega \in \mathcal{D}^n(\Omega \times \mathbb{R}^N)$ by (see [10, Vol. I, Sect. 3.2.1])

$$\begin{aligned} G_u(\omega) &= \int_{\mathcal{G}_{u,\Omega}} \langle \omega, \xi_u \rangle d\mathcal{H}^n \\ &= \int_{\Omega} \langle \omega(x, u(x)), M(Du(x)) \rangle dx \\ &= \sum_{|\alpha|+|\beta|=n} \sigma(\alpha, \bar{\alpha}) \int_{\Omega} \omega_{\alpha\beta}(x, u(x)) M_{\bar{\alpha}}^{\beta}(Du(x)) dx, \end{aligned}$$

where $M(Du)$ is the n -vector in $\bigwedge_n(\mathbb{R}^{n+N})$ given by

$$M(Du) = (e_1 + \sum_{i=1}^N D_1 u^i \epsilon_i) \wedge \dots \wedge (e_n + \sum_{i=1}^N D_n u^i \epsilon_i),$$

$\{e_i\}_{i=1}^n, \{\epsilon_i\}_{i=1}^n$ being canonical basis for \mathbb{R}_x^n , and \mathbb{R}_y^n , respectively.

In order to prove the main result, some lemmas are introduced as follows.

Lemma 3.1. *Let $F : \Omega \rightarrow P(\mathbb{R}^n)$ be a set-valued map. If there exists a constant s such that*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq s|x_1 - x_2|^2$$

for all $(x_1, y_1), (x_2, y_2) \in \Gamma_{F,\Omega}$, then the contingent derivative $DF(x, y)$ at every point $(x, y) \in \Gamma_{F,\Omega}$ is positive definite in the sense that

$$p \cdot q \geq s|p|^2$$

for all $(p, q) \in \Gamma_{DF(x,y), \mathbb{R}^n}$.

Proof. Given $(x, y) \in \Gamma_{F,\Omega}, (p, q) \in \Gamma_{DF(x,y)}$, there exist $h_k \rightarrow 0^+, p_k \rightarrow p$ and $q_k \rightarrow q$ such that

$$(x, y) + h_k(p_k, q_k) \in \Gamma_{F,\Omega}.$$

So $y + h_k q_k \in F(x + h_k p_k)$ and

$$h_k^2 p_k \cdot q_k = \langle y + h_k q_k - y, x + h_k p_k - x \rangle \geq s h_k^2 |p_k|^2,$$

which implies that $p \cdot q \geq s|p|^2$. □

Lemma 3.2. *Let A be an $n \times n$ matrix and semi ($s = 0$), weak ($s < 0$), strongly ($s > 0$) positive definite in the sense that*

$$x^T A x \geq s x^T x$$

for all $x \in \mathbb{R}^n, x \neq 0$. Then $\operatorname{Re}(\lambda) \geq s$, where $\operatorname{Re}(\lambda)$ is the real part of any eigenvalue λ of A . Furthermore, $\det A \geq$ (or $>$) 0 if A is semi (or strongly) positive definite.

Proof. Let $\lambda = \mu + i\nu \in \mathbb{C}$ be an eigenvalue of A with $\mu, \nu \in \mathbb{R}$, and $z \in \mathbb{C}^n$ be a right eigenvector associated with λ . Decompose z as $x + iy$ with $x, y \in \mathbb{R}^n$. Then $(A - \lambda)z = 0$, and thus

$$\begin{cases} (A - \mu)x + \nu y = 0, \\ (A - \mu)y - \nu x = 0, \end{cases}$$

which implies that $x^T(A - \mu)x + y^T(A - \mu)y = \nu(y^Tx - x^Ty) = 0$, and hence

$$\mu = \frac{x^TAx + y^TAy}{x^Tx + y^Ty} \geq s.$$

□

Lemma 3.3. *Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ matrix, then for any $c, d > 0$*

$$\sum_{|\alpha|+|\beta|=n} (M_{\alpha\beta}(A, cI - dA))^2 > 0.$$

Proof. If $A = 0$, then $M_{\alpha\beta}(A, cI - A) \neq 0$ for $\alpha = 0, \beta = (1, 2, \dots, n)$. If $\text{rank}(A) = r > 0$, then there exist $\alpha, \beta \in I(r, n)$ such that $M_{\beta}^{\alpha}(A) \neq 0$. A can be written as $A := (\rho_1, \rho_2, \dots, \rho_n)^T$, where $\rho_i \in \mathbb{R}^n$. Then $\rho_{\alpha_1}, \rho_{\alpha_2}, \dots, \rho_{\alpha_r}$ are linearly independent and form a basis of $\rho_1, \rho_2, \dots, \rho_n$. Let \longrightarrow be reversible linear transformations, then some tedious manipulation yields

$$(\rho_{\alpha_1}, \dots, \rho_{\alpha_r}, cI_{\beta_1} - d\rho_{\beta_1}, \dots, cI_{\beta_{n-r}} - d\rho_{\beta_{n-r}})^T \longrightarrow \begin{pmatrix} \frac{A_{\beta}^{\alpha}}{0} & 0 \\ 0 & I_{\beta} \end{pmatrix},$$

which implies $M_{\alpha\beta}(A, cI - A) \neq 0$. □

Let $F : \Omega \rightarrow P_0(\mathbb{R}^n)$ be a set-valued map such that $F + cI$ is maximal monotone in Ω , then $f : \Omega \rightarrow \mathbb{R}^n$ such that $f(x) \subset F(x)$ for any $x \in \Omega$. According to [1, Theorem 3.2], f is approximately differentiable a.e. with approximate differential Df . Given $s > c$, we define a rotation transformation Θ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$(x', y') \mapsto (\cos \theta x' - \sin \theta y', \sin \theta x' + \cos \theta y'),$$

where $\theta := \arccos \frac{2s}{\sqrt{1+4s^2}} < \arccos \frac{2c}{\sqrt{1+4c^2}} := \theta_0$. With the help of the preceding lemmas we can now prove the following theorem.

Theorem 3.4. *F, f and Θ are given as above. The following hold:*

(i) $\Theta^{-1}(\Gamma_{F,\Omega})$ can be written as the graph of a Lipschitz function u , i.e.,

$$\Gamma_{F,\Omega} = \Theta(\Gamma_{u,D}),$$

where $D := \{\cos \theta x + \sin \theta y \mid (x, y) \in \Gamma_{F,\Omega}\}$.

(ii) $\Theta_{\#}G_u = \tau(\Gamma_{F,\Omega}, 1, \xi)$.

(iii) $\xi(x, f(x)) = \xi_f(x, f(x))$ for \mathcal{L}^n a.e. $x \in \Omega$, where $\xi_f(x, f(x)) := M(Df(x))/|M(Df(x))|$.

(iv) If $0 < \theta_1 < \theta_2 < \theta_0$, then $\Theta_{1\#}G_{u_1} = \Theta_{2\#}G_{u_2}$.

Proof. It is simple to show that there exists $l > 0$ such that

$$|y'_1 - y'_2|^2 \leq l|x'_1 - x'_2|^2 \quad \text{for any } (x'_1, y'_1), (x'_2, y'_2) \in \Theta^{-1}(\Gamma_{F,\Omega}),$$

where $l \geq \max\{4s^2, \frac{4sc+1}{4s^2-4sc}\}$, which implies (i). Moreover, we can show that D is a domain in \mathbb{R}^n and

$$\text{Lip}(u, D) := \sup \left\{ \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|} \mid x_1, x_2 \in D, x_1 \neq x_2 \right\} \leq \max \left\{ 2s, \sqrt{\frac{4sc+1}{4s^2-4sc}} \right\}.$$

Then the i.m. rectifiable current $G_u = \tau(\Gamma_{u,D}, 1, \xi_u)$ is carried by the graph of u .

An easy deduction gives that

$$\langle u(x'_1) - u(x'_2), x'_1 - x'_2 \rangle \leq 2s|x'_1 - x'_2|^2 \text{ for any } x'_1, x'_2 \in D. \quad (3.1)$$

For any $x' \in D$, let $\mathcal{M} := \Gamma_{u,D}$ and $A := (a_{ij})_{n \times n} = Du(x')$, then

$$M(Du(x')) = (e'_1 + \sum_{s=1}^n a_{s1}\epsilon'_s) \wedge \cdots \wedge (e'_n + \sum_{s=1}^n a_{sn}\epsilon'_s).$$

Let $\tau_i := e'_i + \sum_{s=1}^n a_{si}\epsilon'_s$ and $\zeta_i := d^{\mathcal{M}}\Theta(\tau_i)$. Note that

$$\begin{aligned} \zeta_i &= \sum_{s=1}^n (\tau_i \cdot \nabla^{\mathcal{M}} \Theta^s) e_s + \sum_{s=1}^n (\tau_i \cdot \nabla^{\mathcal{M}} \Theta^{n+s}) \varepsilon_s \\ &= \sum_{j=1}^n (\cos \theta \delta_{ji} - \sin \theta a_{ji}) e_j + \sum_{j=1}^n (\sin \theta \delta_{ji} + \cos \theta a_{ji}) \varepsilon_j. \end{aligned}$$

Set $P = (\cos \theta I - \sin \theta A)$ and $Q = (\sin \theta I + \cos \theta A)$. Then $Q = \sqrt{(1 + 4c^2)}I - 2cP$ and

$$\begin{aligned} \zeta_1 \wedge \cdots \wedge \zeta_n &= \left(\sum_{j=1}^n p_{j1} e_j + \sum_{j=1}^n q_{j1} \varepsilon_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n p_{jn} e_j + \sum_{j=1}^n q_{jn} \varepsilon_j \right) \\ &= \sum_{|\alpha|+|\beta|=n} \sum_{|\gamma|=|\beta|} \sigma(\bar{\gamma}, \gamma) M_{\bar{\gamma}}^{\alpha}(P) M_{\gamma}^{\beta}(Q) e_{\alpha} \wedge \varepsilon_{\beta} \\ &= \sum_{|\alpha|+|\beta|=n} \sigma(\bar{\beta}, \beta) M_{\alpha\beta}(P, Q) e_{\alpha} \wedge \varepsilon_{\beta}. \end{aligned}$$

By Lemma 3.3

$$\begin{aligned} \mathcal{M}_+ &= \{(x', y') \in \mathcal{M} \mid |(\wedge_n d^{\mathcal{M}} \Theta)(M(Du(x')))| > 0\} \\ &= \{(x', u(x')) \in \mathcal{M} \mid |\zeta_1 \wedge \cdots \wedge \zeta_n| > 0\} \\ &= \{(x', u(x')) \in \mathcal{M} \mid Du(x') \text{ exist}\}, \end{aligned}$$

which implies that $\mathcal{H}^n(\mathcal{M} \setminus \mathcal{M}^+) = 0$. According to (2.1), it follows that for any $\omega(x, y) \in \mathcal{D}^n(\Omega \times \mathbb{R}^n)$,

$$\begin{aligned} \Theta_{\#} G_u(\omega(x, y)) &= \int_{\Gamma_{F,\Omega}} \langle \omega(x, y), \frac{\zeta_1 \wedge \cdots \wedge \zeta_n}{|(\zeta_1 \wedge \cdots \wedge \zeta_n)|} (\cos \theta x + \sin \theta y, -\sin \theta x + \cos \theta y) \rangle d\mathcal{H}^n(x, y) \\ &= \tau(\Gamma_{F,\Omega}, 1, \xi)(\omega(x, y)), \end{aligned}$$

where the orientation $\xi(x, y) = \frac{\zeta_1 \wedge \cdots \wedge \zeta_n}{|(\zeta_1 \wedge \cdots \wedge \zeta_n)|} (\cos \theta x + \sin \theta y, -\sin \theta x + \cos \theta y)$ for \mathcal{H}^n -a.e. $(x, y) \in \Gamma_{F,\Omega}$. Therefore $\Theta_{\#} G_u = \tau(\Gamma_{F,\Omega}, 1, \xi)$.

Set $E := \{x \in \Omega \mid x \in \mathcal{L}_f, Df(x), Du(\cos \theta x + \sin \theta f(x)) \text{ exists}\}$. According to Proposition 2.5 (iii) and the fact that $u \in L(D)$, it follows that $\mathcal{L}^n(\Omega \setminus E) = 0$.

Fix $x_0 \in E$ and denote $A := Du(\cos \theta x_0 + \sin \theta f(x_0))$, $B := Df(x_0)$. Since $u(\cos \theta x_0 + \sin \theta f(x_0)) = -\sin \theta x_0 + \cos \theta f(x_0)$, then

$$A(\cos \theta I + \sin \theta B) = -\sin \theta I + \cos \theta B, \quad (\cos \theta A + \sin \theta I) = (\cos \theta I - \sin \theta A)B.$$

So

$$(\cos \theta I - \sin \theta A)(\cos \theta I + \sin \theta B) = I,$$

which implies that $(\cos \theta I - \sin \theta A)$ is reversible and $(\sin \theta I + \cos \theta A)(\cos \theta I - \sin \theta A)^{-1} = B$. Let $P = (\cos \theta I - \sin \theta A)$ and $Q = (\sin \theta I + \cos \theta A)$ for convenience, then

$$\begin{aligned} & \zeta_1 \wedge \cdots \wedge \zeta_n (\cos \theta x_0 + \sin \theta f(x_0), -\sin \theta x_0 + \cos \theta f(x_0)) \\ &= \left(\sum_{j=1}^n p_{j1} e_j + \sum_{j=1}^n q_{j1} \varepsilon_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n p_{jn} e_j + \sum_{j=1}^n q_{jn} \varepsilon_j \right) \\ &= \left(\sum_{j=1}^n p_{j1} e_j + \sum_{j=1}^n (BP)_{j1} \varepsilon_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^n p_{jn} e_j + \sum_{j=1}^n (BP)_{jn} \varepsilon_j \right) \\ &= \det(P^T) (e_1 + \sum_{s=1}^n b_{s1} \varepsilon_s) \wedge \cdots \wedge (e_n + \sum_{s=1}^n b_{sn} \varepsilon_s). \end{aligned}$$

Note that $\det P \geq 0$ by (3.1), Lemma 3.1 and Lemma 3.2, and thus

$$\begin{aligned} \xi(x_0, f(x_0)) &= \text{sign}(P) \frac{(e_1 + \sum_{s=1}^n b_{s1} \varepsilon_s) \wedge \cdots \wedge (e_n + \sum_{s=1}^n b_{sn} \varepsilon_s)}{|(e_1 + \sum_{s=1}^n b_{s1} \varepsilon_s) \wedge \cdots \wedge (e_n + \sum_{s=1}^n b_{sn} \varepsilon_s)|} \\ &= M(Df(x_0)) / |M(Df(x_0))|. \end{aligned}$$

Therefore $\xi_f(x, f(x)) = \xi(x, f(x))$ for \mathcal{L}^n a.e. $x \in \Omega$.

If $0 < \theta_1 < \theta_2 < \theta_0$ with $\theta_i := \arccos \frac{2s_i}{\sqrt{1+4s_i^2}}$, $i=1,2$. Then there exists a transformation Θ_3 such that $\Theta_3(\Gamma_{u_2, D_2}) = \Gamma_{u_1, D_1}$. In order not to confuse matters, we write

$$\Theta_1 : \begin{cases} x = \cos \theta_1 x' - \sin \theta_1 y' \\ y = \sin \theta_1 x' + \cos \theta_1 y' \end{cases}, \quad \Theta_2 : \begin{cases} x = \cos \theta_2 x'' - \sin \theta_2 y'' \\ y = \sin \theta_2 x'' + \cos \theta_2 y'' \end{cases}, \quad \Theta_3 : \begin{cases} x' = \cos \theta_3 x'' - \sin \theta_3 y'' \\ y' = \sin \theta_3 x'' + \cos \theta_3 y'' \end{cases}.$$

Clearly, $\Theta_2 = \Theta_1 \circ \Theta_3$ and $\theta_2 = \theta_1 + \theta_3$.

Let $H(x'') = (\cos \theta_3 I - \sin \theta_3 u_2)(x'')$ where $x'' \in D_2$. Some tedious manipulation yields that there exists $l > 0$ such that

$$\langle H(x''_1) - H(x''_2), x''_1 - x''_2 \rangle \geq l |x''_1 - x''_2|^2 \quad (3.2)$$

for any $x''_1, x''_2 \in D_2$. For any x''_0 such that $Du_2(x''_0)$ exists, it follows from Lemma 3.1 and (3.2) that

$$p^T DH(x''_0, H(x''_0))p = p^T (\cos \theta_3 I - \sin \theta_3 Du_2(x''_0))p \geq lp^T p$$

for all $p \in \mathbb{R}^n$, which implies $\det(\cos \theta_3 I - \sin \theta_3 Du_2(x''_0)) > 0$. Then an argument similar to the one as above shows that

$$G_{u_1} = \Theta_{3\#} G_{u_2}.$$

Hence

$$\Theta_{2\#} G_{u_2} = \Theta_{1\#} \circ \Theta_{3\#} G_{u_2} = \Theta_{1\#} G_{u_1},$$

which completes the proof. \square

Definition 3.5. Let $F : \Omega \rightarrow P_0(\mathbb{R}^n)$ be a maximal semi-monotone map, we define the Cartesian current G_F associated to F as

$$G_F := \Theta_{\#} G_u,$$

where Θ, u are given in Theorem 3.4.

This quantity is well-defined since G_F is independent of the rotation transformations and the orientation of the current is consistent with the one defined in the class $\mathcal{A}^1(\Omega, \mathbb{R}^n)$.

Proof of Theorem 1.1. The integer multiplicity rectifiable n -current $G_{\partial w}$ carried by $\Gamma_{\partial w, \Omega}$ is defined by Definition 3.5. For any open set $\Omega' \subset\subset \Omega$, $D' := \pi \circ \Theta(\Gamma_{\partial w, \Omega'})$ is bounded since $\partial w(\Omega')$ is bounded. Then a tedious computation implies that

$$\mathcal{H}^n(\Gamma_{\partial w, \Omega'}) = \mathbf{M}_{\Omega' \times \mathbb{R}^n}(G_{\partial w}) = \mathbf{M}_{D' \times \mathbb{R}^n}(G_u) = \int_{D'} |M(Du)| dx'. \quad (3.3)$$

So (i), (ii) can be easily proved by Theorem 3.4 and (3.3).

Given $\eta(x, y) \in \mathcal{D}^{n-1}(\Omega \times \mathbb{R}^n)$, let $\Theta(D \times \mathbb{R}^n) = U$, then $U \cap (\Omega \times \mathbb{R}^n)$ is an open set in $\Omega \times \mathbb{R}^n$. And thus $\text{spt } \eta \cap \text{spt } G_{\partial w}$ is compact in $U \cap (\Omega \times \mathbb{R}^n)$. So there exists $\zeta \in C_0^\infty(U \cap (\Omega \times \mathbb{R}^n))$ such that $\zeta = 1$ in a neighbourhood of $\text{spt } \eta \cap \text{spt } G_{\partial w}$. Thus

$$\partial G_{\partial w}(\eta) = G_{\partial w}(d\eta) = G_{\partial w}(\zeta d\eta) = G_{\partial w}(d(\zeta\eta)) = \partial G_u(\Theta^\sharp(\zeta\eta)) = 0,$$

where the last equality follows from $\Theta^\sharp(\zeta\eta) \in \mathcal{D}^{n-1}(D \times \mathbb{R}^n)$. So $\partial G_{\partial w} \llcorner \Omega \times \mathbb{R}^n = 0$. \square

4 The proof of Theorem 1.2

Lemma 4.1. *Let $u, \{u_k\}_{k=1}^\infty \subset L(\Omega, \mathbb{R}^n)$ such that $\text{Lip}(u_k), \text{Lip}(u)$ uniformly bounded and u_k converge uniformly to u in Ω . Then*

$$M_\alpha^\beta(Du_k) \rightharpoonup M_\alpha^\beta(Du) \text{ in } L^1(\Omega)$$

for any ordered multi-indices α, β with $|\alpha| + |\beta| = n$.

Proof. Note that Laplace's formulas yield

$$\begin{aligned} M_\alpha^\beta(Du) &= \sum_{j \in \bar{\alpha}} D_j u^i ((\text{adj } Du)_{\bar{\alpha}}^\beta)_j^i \\ &= \sum_{j \in \bar{\alpha}} \sigma(i, \beta - i) \sigma(j, \bar{\alpha} - j) M_{\bar{\alpha} - j}^{\beta - i}(Du) D_j u^i. \end{aligned}$$

u is a Lipschitz function which implies

$$\sum_{j \in \bar{\alpha}} D_j ((\text{adj } Du)_{\bar{\alpha}}^\beta)_j^i = 0$$

in the sense of distribution. So for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_U M_\alpha^\beta(Du) \varphi dx' = - \int_U u^i \sum_{j \in \bar{\alpha}} D_j \varphi ((\text{adj } Du)_{\bar{\alpha}}^\beta)_j^i dx'.$$

Since Du_k are uniformly Lipschitz functions and $u_k \rightrightarrows u$ in Ω , in order to prove the weak convergence of minors in L^1 it suffices to show that for all $\varphi \in C_c^\infty(\Omega)$.

$$\int_U \varphi M_\alpha^\beta(Du_k) dx' \rightarrow \int_U \varphi M_\alpha^\beta(Du) dx' \quad (4.1)$$

We shall now prove (4.1) by induction on the order of the minors. Obviously it holds for $l = 1$ since $u_k \rightrightarrows u$ in U . Suppose that it holds for $l - 1$. Clearly,

$$\begin{aligned} \int_U \varphi M_{\alpha}^{\beta}(Du_k) dx' &= - \int_U u_k^i \sum_{j \in \bar{\alpha}} D_j \varphi ((\text{adj } Du_k)_{\alpha}^{\beta})_j^i dx' \\ &= - \int_U u^i \sum_{j \in \bar{\alpha}} D_j \varphi ((\text{adj } Du_k)_{\alpha}^{\beta})_j^i dx' + \int_U (u_i - u_k^i) \sum_{j \in \bar{\alpha}} D_j \varphi ((\text{adj } Du_k)_{\alpha}^{\beta})_j^i dx'. \end{aligned}$$

By the inductive assumption the first integral on the right tends to

$$- \int_U u^i \sum_{j \in \bar{\alpha}} D_j \varphi ((\text{adj } Du)_{\alpha}^{\beta})_j^i dx',$$

which is equal to

$$\int_U \varphi M_{\alpha}^{\beta}(Du) dx'.$$

While the second integral on the right tends to 0, this proves (4.1) for l and therefore the theorem. \square

Proof of Theorem 1.2. It suffices to prove the theorem in the case that w_k, w are Lipschitz in Ω . According to [5, Lemma 2.3.] and [11, Theorem B.3.1.4], it follows that w_k, w can be extended to be semi-convex functions as w_k^*, w^* , defined in \mathbb{R}^n , such that $w_k^* \rightrightarrows w^*$ in \mathbb{R}^n . Let $s > c$, and note that $(\cos \theta I + \sin \theta \partial w_k^*)(\mathbb{R}^n) = \mathbb{R}^n$ where $\cos \theta = \frac{2s}{\sqrt{1+4s^2}}$. then by the proof of Theorem 3.4, there exist a rotation transformation Θ and Lipschitz functions $u_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\Theta_{\#} G_{u_k} = G_{\partial w_k^*}, \quad \text{Lip}(u_k) \leq \max\{2s, \sqrt{\frac{4sc+1}{4s^2-4sc}}\},$$

where $k = 0, 1, 2, \dots$ and $w_0^* := w^*$.

First, we need to show that $u_k \rightrightarrows u$ in any compact $K \subset \mathbb{R}^n$. This result will follow from Arzela-Ascoli Theorem, if we can show that $u_k(x') \rightarrow u(x')$ for any $x' \in \mathbb{R}^n$. Here we argue by contradiction, assume that there exists x' such that $u_k(x') \not\rightarrow u(x')$. Let

$$x_k := \cos \theta x' - \sin \theta u_k(x'), \quad y_k := \sin \theta x' + \cos \theta u_k(x').$$

Then $x' = \cos \theta x_k + \sin \theta y_k$. and thus $y_k - y_0 = -2s(x_k - x_0)$. Note that $y_k + cx_k \in \partial v_k^*(x_k)$ where $v_k^*(x) := w_k^*(x) + c|x|^2$ are convex and Lipschitz in \mathbb{R}^n . Hence

$$(2s - c)x_k = -(y_k + cx_k) - 2sx_0 + y_0,$$

which implies that both x_k and y_k are bounded.

If $x_k \not\rightarrow x_0$, there exist $\epsilon_0 > 0$ and a subsequence x_{λ_k} such that $|x_{\lambda_k} - x_0| \geq \epsilon_0$. Since x_{λ_k} and y_{λ_k} are bounded, then there exists a subsequence x_{μ_k} of x_{λ_k} such that

$$x_{\mu_k} \rightarrow x_1, \quad y_{\mu_k} \rightarrow y_1.$$

By using Proposition 2.5 (ii),

$$w_{\mu_k}^*(z) \geq w_{\mu_k}^*(x_{\mu_k}) + \langle y_{\mu_k}, z - x_{\mu_k} \rangle - \frac{c}{2} |z - x_{\mu_k}|^2$$

for any $z \in \mathbb{R}^n$. Since $w_k^* \rightrightarrows w^*$ in \mathbb{R}^n , it follows that

$$w^*(z) \geq w^*(x_1) + \langle y_1, z - x_1 \rangle - \frac{c}{2}|z - x_1|^2,$$

which implies $y_1 \in \partial w^*(x_1)$. So $x' = \cos \theta x_1 + \sin \theta y_1$, and then

$$\begin{aligned} 0 &= |x' - x'|^2 = (\cos \theta(x_1 - x_0) + \sin \theta(y_1 - y_0))^2 \\ &\geq \frac{1}{1 + 4s^2}(4s^2 - 4sc)(x_1 - x_0)^2 \\ &\geq 0. \end{aligned}$$

Hence $x_1 = x_0$ which contradicts the assumption that $|x_{\mu_k} - x_0| \geq \epsilon_0$.

In order to prove the Theorem, according to the fact that $G_{\partial w_k^*} \lrcorner \Omega \times \mathbb{R}^n = G_{\partial w_k}$, it is enough to show that $G_{\partial w_k^*} \rightharpoonup G_{\partial w^*}$ in $\mathcal{D}^n(\mathbb{R}^n \times \mathbb{R})$. Since $\Theta_{\sharp} G_{u_k} = G_{\partial w_k^*}$, we need only to show that for any ordered multi-indices α, β with $|\alpha| + |\beta| = n$ and $\varphi(x', y') \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$

$$\int \varphi(x', u_k(x')) M_{\alpha}^{\beta}(Du_k(x')) dx' \rightarrow \int \varphi(x', u(x')) M_{\alpha}^{\beta}(Du(x')) dx',$$

which can be deduced by Lemma 4.1 and the fact that $u_k \rightrightarrows u$ in any compact $K \subset \mathbb{R}^n$. \square

The following comes easily from the standard mollification of $w \in W(\Omega)$.

Corollary 4.2. *If $w \in W(\Omega, c)$, then there exists a sequence $\{\omega_k\}_{k=1}^\infty \subset W(\Omega', c) \cap C^\infty(\Omega', \mathbb{R}^n)$ such that $G_{\partial \omega_k} \rightharpoonup G_{\partial w}$ in $\mathcal{D}^n(\Omega' \times \mathbb{R}^n)$ for any open set $\Omega' \subset \subset \Omega$. Moreover, it holds for Ω if ω is Lipschitz.*

Corollary 4.3. *If $w \in W(\Omega)$ and $n \geq 2$, then $G_{\partial \omega}$ is Lagrangian.*

Proof. It suffices to prove the theorem in the case that ω is Lipschitz in Ω . Then there exists a sequence $\{\omega_k\}_{k=1}^\infty \subset W(\Omega) \cap C^\infty(\Omega, \mathbb{R}^n)$ such that $G_{\partial \omega_k} \rightharpoonup G_{\partial \omega}$. For any $\eta \in \mathcal{D}_{n-2}(\Omega \times \mathbb{R}^n)$, since $D^2 \omega_k = (D^2 \omega_k)^T$,

$$G_{\partial \omega_k}(\phi \wedge \eta) = 0,$$

where $\phi := \sum_{i=1}^\infty dx^i \wedge dy^i$. Which proves $G_{\partial \omega}(\phi \wedge \eta) = 0$ and therefore the theorem. \square

5 The proof of Theorem 1.3.

Proof. Without loss of generality, we may assume that w is Lipschitz in Ω . Given $s > c$, there exist a rotation transformation Θ and a Lipschitz function $u : D \rightarrow \mathbb{R}^n$ such that $\Theta_{\sharp} G_u = G_{\partial w}$, where $\theta := \arccos \frac{2s}{\sqrt{1+4s^2}}$. We denote $g_1(x, y) = \cos \theta x + \sin \theta y$, $f_2(x', y') = \sin \theta x' + \cos \theta y'$, and $D_B = \{\cos \theta x + \sin \theta y \mid x \in B, y \in \partial w(x)\}$, where B is a Borel subset in Ω .

On the one hand,

$$g_{1\sharp} G_{\partial w} = g_{1\sharp} \Theta_{\sharp} G_u = \pi_{\sharp} G_u = [D].$$

On the other hand, for any $\varphi \in C_c^\infty(D)$

$$\begin{aligned} g_{1\sharp} G_{\partial w}(\varphi(x') dx') &= G_{\partial w}(\varphi \circ g_1(x, y) dg_1) \\ &= G_{\partial w}(\varphi \circ g_1(x, y) \sum \sigma(\alpha, \bar{\alpha}) \cos^{|\alpha|} \theta \sin^{|\bar{\alpha}|} \theta dx^\alpha \wedge dy^{\bar{\alpha}}) \end{aligned}$$

$$= \sum_{i=0}^n \cos^i \theta \sin^{n-i} \theta \sum_{|\alpha|=i} \sigma(\alpha, \bar{\alpha}) G_{\partial w}^{\alpha \bar{\alpha}}(\varphi \circ g_1(x, y)).$$

Hence

$$[D](\varphi(x')dx') = \sum_{i=0}^n \cos^i \theta \sin^{n-i} \theta \sum_{|\alpha|=i} \sigma(\alpha, \bar{\alpha}) G_{\partial w}^{\alpha \bar{\alpha}}(\varphi \circ g_1(x, y)).$$

Next, we have to show that for any ordered multi-indices α ,

$$\lim_{\epsilon \rightarrow 0} G_{\partial w}^{\alpha \bar{\alpha}}(\varphi_\epsilon \circ g_1(x, y)) = \int_{\Gamma_{\partial w, B}} \xi^{\alpha \bar{\alpha}}(x, y) d\mathcal{H}^n(x, y),$$

where $\varphi_\epsilon = \phi_\epsilon * \chi_{D_B}$. Clearly, $\varphi_\epsilon \in C_c^\infty(D)$ and $\varphi_\epsilon \rightarrow \varphi := \chi_{D_B}$ a.e., as $\epsilon \rightarrow 0$.

In order to prove the claim it suffices to show that

$$\nu_\epsilon(x, y) \rightarrow \nu(x, y) \quad \mathcal{H}^n \text{ a.e. } (x, y) \in \mathbb{R}^{2n},$$

where $\nu_\epsilon(x, y) = \chi_{\Gamma_{\partial w, \Omega}} \varphi_\epsilon(\cos \theta x + \sin \theta y) \xi^{\bar{\alpha} \alpha}(x, y)$ and $\nu(x, y) = \chi_{\Gamma_{\partial w, \Omega}} \varphi(\cos \theta x + \sin \theta y) \xi^{\bar{\alpha} \alpha}(x, y)$.

If $(x, y) \notin \Gamma_{\partial w, \Omega}$, then $\nu_\epsilon(x, y) = 0 = \nu(x, y)$. If $(x, y) \in \Gamma_{\partial w, \Omega}$, there exists $S \subset \Gamma_{\partial w, \Omega}$ such that $\mathcal{H}^n(S) > 0$ and $\nu_\epsilon(x, y) \rightarrow \nu(x, y)$ for any $(x, y) \in S$. Let $S' = g_1(S)$, then

$$\mathcal{H}^n(S) = \mathcal{H}^n(\Gamma_{u, S'}) = \int_{S'} |M(Du)| dx' > 0.$$

Therefore $\mathcal{H}^n(S') > 0$, which contradicts the assumption that $\varphi_\epsilon \rightarrow \varphi$ a.e. Hence the desired result is obtained by the dominated convergence theorem. So

$$\begin{aligned} \mathcal{L}^n(D_B) &= \lim_{\epsilon \rightarrow 0} [D](\varphi_\epsilon(x')dx') \\ &= \lim_{\epsilon \rightarrow 0} \sum_{i=0}^n \cos^i \theta \sin^{n-i} \theta \sum_{|\alpha|=i} \sigma(\alpha, \bar{\alpha}) G_{\partial w}^{\alpha \bar{\alpha}}(\varphi_\epsilon \circ g_1(x, y)) \\ &= \sum_{i=0}^n \cos^i \theta \sin^{n-i} \theta \sum_{|\alpha|=i} \sigma(\alpha, \bar{\alpha}) \int_{\Gamma_{\partial w, B}} \xi^{\alpha \bar{\alpha}}(x, y) d\mathcal{H}^n(x, y). \end{aligned}$$

By the Steiner formula for semi-convex in [7] we have

$$F_k(w, B) = \sum_{|\alpha|=k} \sigma(\alpha, \bar{\alpha}) \int_{\Gamma_{\partial w, B}} \xi^{\alpha \bar{\alpha}}(x, y) d\mathcal{H}^n(x, y).$$

In particular,

$$\begin{aligned} F_0(w, B) &= \lim_{\epsilon \rightarrow 0} G_{\partial w}^{\bar{0}0}(\varphi_\epsilon \circ g_1(x, y)) \\ &= \lim_{\epsilon \rightarrow 0} [D](\varphi_\epsilon(x')d(\cos \theta x' - \sin \theta u(x'))) \\ &= \int_{D_B} \det(\cos \theta I - \sin \theta Du) dx' \\ &= \mathcal{L}^n(B). \end{aligned}$$

If w is convex,

$$\begin{aligned}
F_n(w, B) &= \lim_{\epsilon \rightarrow 0} G_{\partial w}^{00}(\varphi_\epsilon \circ g_1(x, y)) \\
&= \lim_{\epsilon \rightarrow 0} [D](\varphi_\epsilon(x') d(\sin \theta x' + \cos \theta u(x'))) \\
&= \int_{D_B} \det(\sin \theta I + \cos \theta Du) dx' \\
&= \int_{f_2(D_B)} \mathcal{H}^0(D_B \cap f_2^{-1}(y)) dy,
\end{aligned}$$

where the last equality is deduced by area formula and the fact $\det(\sin \theta I + \cos \theta Du) \geq 0$. Let $P := \{y \in f_2(D_B) \mid \mathcal{H}^0(D_B \cap f_2^{-1}(y)) \neq 1\}$, and fix $y \in P$. Then there exist $x_1, x_2 \in B$ such that $x_1 \neq x_2$ and $y \in \partial w(x_1) \cap \partial w(x_2)$, and hence $\mathcal{H}^n(P) = 0$ by the proof of Theorem 5.11 in [1]. Therefore

$$F_n(w, B) = \mathcal{L}^n(f_2(D)) = \mathcal{L}^n(\partial w(B)),$$

which completes the proof. \square

Remark 5.1. It should be observed that the measures $C_n^k F_k$ in the notation of Colesanti-Hug correspond to the Hessian measures F_k in the notation of Trudinger-Wang, and in this paper we denote F_k in the same way as the latter.

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